

MULTIGRAPH HYPERPLANE ARRANGEMENTS AND PARKING FUNCTIONS.

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ABSTRACT. Back in the nineties Pak and Stanley introduced a labeling of the regions of a k -Shi arrangement by k -parking functions and proved its bijectivity. Duval, Klivans, and Martin considered a modification of this construction associated with a graph G . They introduced the G -Shi arrangement and a labeling of its regions by G -parking functions. They conjectured that their labeling is surjective, i.e. that every G -parking function appears as a label of a region of the G -Shi arrangement. Later Hopkins and Perkinson proved this conjecture. In particular, this provided a new proof of the bijectivity of Pak-Stanley labeling in the $k = 1$ case. We generalize Hopkins-Perkinson's construction to the case of arrangements associated with oriented multi-graphs. In particular, our construction provides a simple straightforward proof of the bijectivity of the original Pak-Stanley labeling for arbitrary k .

INTRODUCTION

Shi arrangement were introduced by Shi in [3] in his study of Kazhdan-Lusztig polynomials and cells of affine symmetric groups. In [4] Stanley studied the enumeration of the connected components of the complement to Shi arrangements (regions) with respect to distance from a base region, where the distance between two regions is defined as the number of hyperplanes of the arrangement separating them. The key step in Stanley's construction is a bijection between the set of regions and the set of parking functions, such that the sum of values of a parking function is equal to the distance from the corresponding region to the base region. This bijection is usually referred to as Pak-Stanley labeling. The construction of Pak and Stanley also applies to the extended Shi arrangements, also called k -Shi arrangements. The regions of k -Shi arrangements are labeled by k -parking functions.

The original proof of bijectivity of the Pak-Stanley labeling is very complicated. The objective of this paper is to generalize the labeling to a larger class of arrangements and to prove surjectivity of the generalized labeling (with an appropriately generalized notion of parking functions). In the case of k -Shi arrangements, the bijectivity follows then from comparing the number of regions and the number parking functions (both are computed in fairly elementary way). This provides a simple and straightforward proof of the bijectivity of the original Pak-Stanley labeling.

Our constructions were inspired by work of Hopkins and Perkinson ([2]), where they prove surjectivity of Pak-Stanley labeling for arrangements associated with a graph. This was originally conjectured by Duval, Klivans, and Martin ([1]). Hopkins-Perkinson's results imply bijectivity of the original Pak-Stanley labeling for $k = 1$. We extend their construction in order to cover the case of general $k \geq 1$. We also simplify the key step in their proof by replacing it with a transparent geometric argument.

The rest of the paper is organized as follows. We begin by reviewing the original Pak-Stanley labeling in Section 1. In Section 2 we review the generalization associated with a graph. Finally, in Section 3 we present our construction and prove the surjectivity of the labeling.

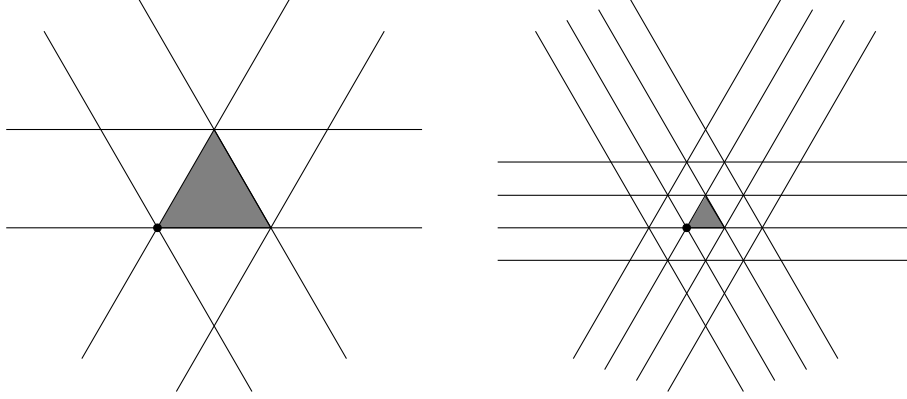


FIGURE 1. Here $n = 3$. 1-Shi arrangement on the left and 2-Shi arrangement on the right.

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1. PAK-STANLEY LABELING

Let $V := \{x_1 + \dots + x_n = 0\} \subset \mathbb{R}^n$. We will use the following notation: for any $i, j \in \{1, \dots, n\}$ and $a \in \mathbb{R}$, let

$$H_{ij}^a := \{x_i - x_j = a\} \subset V.$$

Definition 1.1. The *affine braid arrangement*: $\tilde{B}_n := \{H_{ij}^l : 0 < j < i \leq n, l \in \mathbb{Z}_{\geq 0}\}$ is the set of all such hyperplanes with integer right hand side. Connected components of the complement to \tilde{B}_n are called *alcoves*. The *fundamental alcove*: is given by

$$A_0 := \{\bar{x} \in V \mid x_1 > x_2 > \dots > x_n > x_1 - 1\}$$

Extended Shi arrangements are defined as subarrangements in \tilde{B}_n :

Definition 1.2. The *k-Shi arrangement* is given by

$$Sh_n^k := \{H_{ij}^l : 0 < j < i \leq n, -k < l \leq k\}$$

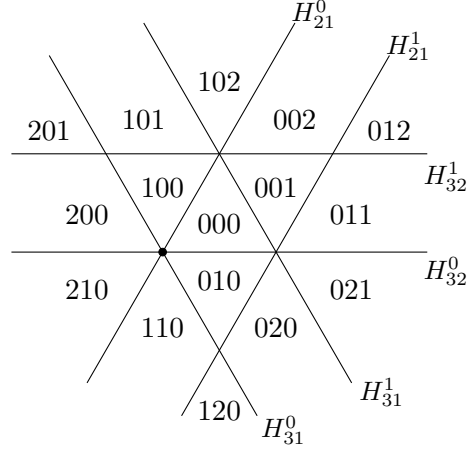
See Figure 1 for examples.

Definition 1.3. A function $f : \{1, \dots, n\} \rightarrow \mathbb{Z}_{\geq 0}$ is called a *k-parking function* if for any subset $I \subset \{1, \dots, n\}$ there exist a number $i \in I$ such that $f(i) \leq k(n - \#I)$

Remark 1. Alternatively, one can define *k-parking functions* as follows. Consider the Young diagram with row lengths given by the values $f(1), \dots, f(n)$ put in non-increasing order. Then f is a *k-parking function* if and only if the diagram fits under the diagonal of an $n \times kn$ rectangle.

Pak-Stanley labeling is a map from the set of regions of a *k-Shi arrangement* to the set of *k-parking functions*. It can be defined inductively using the following rules:

- (1) Label the fundamental alcove by the zero function.

FIGURE 2. Pak-Stanley labeling for $n = 3$, $k = 1$.

- (2) When crossing a hyperplane H_{ij}^l with $l > 0$ in positive direction, increase the value $f(i)$ by 1.
- (3) When crossing a hyperplane H_{ij}^l with $l \leq 0$ in positive direction, increase the value $f(j)$ by 1.

Here “in positive direction” means getting further away from the fundamental region. More directly, the parking function λ_R corresponding to a region R is given by the following formula:

$$\lambda_R(a) = \# \{H_{ij}^l \in H_R : l > 0, i = a\} + \# \{H_{ij}^l \in H_R : l \leq 0, j = a\},$$

where $H_R \subset Sh_n^k$ is the set of hyperplanes separating R from A_0 .

See Figure 2 for the Pak-Stanley labeling for $n = 3$ and $k = 1$.

Theorem 1.1 ([4]). *The Pak-Stanley labelling is a bijection between the set of regions of the k -Shi arrangement and the set of k -parking functions.*

The original proof is complicated, especially for $k > 1$.

2. G -PARKING FUNCTIONS AND G -SHI ARRANGEMENTS

Let G be a graph with the vertex set $\{1, \dots, n\}$. The G -Shi arrangement is the subarrangement in 1-Shi arrangement given by

$$\{H_{ij}^l \in Sh_n^1 : (i, j) \in E_G\}.$$

The Pak-Stanley labeling is defined in the same way as before. See Figure 3 for an example. Note that the labeling is not injective: for example, on Figure 3 the labels in bold are the same. It is not hard to show that the labels satisfy the following condition:

Definition 2.1. A function $f : \{1, \dots, n\} \rightarrow \mathbb{Z}_{\geq 0}$ is called a G -parking function if for any non-empty subset $I \subset \{1, \dots, n\}$ there exists a vertex $i \in I$ such that the number of edges $(i, j) \in E_G$ such that $j \notin I$ is greater than or equal to $f(i)$.

Theorem 2.1 ([2]). *The labeling is a surjective map from the set of regions of a G -Shi arrangement to the set of G -parking functions.*

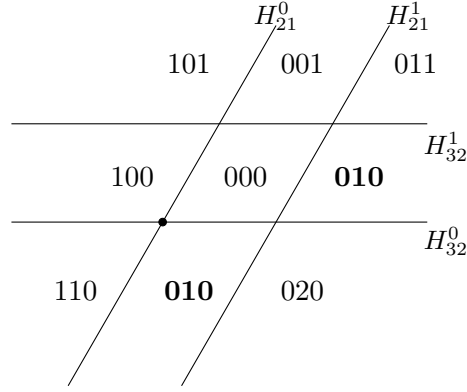


FIGURE 3. Pak-Stanley labeling for the graph $G := \{1 - 2 - 3\}$.

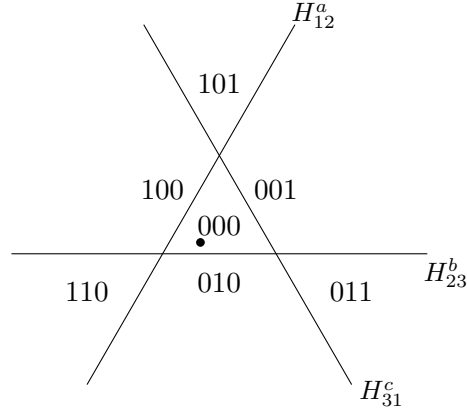


FIGURE 4. Here $a, b, c > 0$, and so $m_{12} = m_{23} = m_{31} = 1$, $m_{32} = m_{21} = m_{13} = 0$, and the dot corresponds to the origin.

This was first conjectured by Duval, Klivans, and Martin ([1]) and then proved by Hopkins and Perkinson. Note that this provides a new proof of bijectivity of the original Pak-Stanley labeling for $k = 1$.

Remark 2. In fact, Hopkins and Perkinson proved more than that. It turns out that one can move the hyperplanes of the arrangement by changing the constant on the right hand side. The labeling will remain surjective as long as the fundamental region does not collapse.

3. MULTIGRAPHICAL GENERALIZATIONS

Let \mathcal{A} be any finite arrangement of hyperplanes of the form $H_{ij}^a = \{x_i - x_j = a\}$ with $a > 0$. The region R_0 containing the origin is called *fundamental region*. Let m_{ij} denote the number of hyperplanes of the form H_{ij}^a in \mathcal{A} . Note that we have $m_{ij} + m_{ji}$ parallel hyperplanes, m_{ij} on one side of the origin and m_{ji} on the other. The labeling defined in the same way as before. In fact, it is more symmetric in our notations. Let R be a region, then $\lambda_R(i)$ equals to the number of hyperplanes H_{ij}^l separating R from R_0 . See Figure 4 for an example.

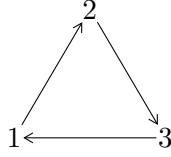


FIGURE 5. The graph corresponding to the arrangement in Figure 4.

The oriented multigraph $G = G_{\mathcal{A}}$ is defined as follows. The vertex set is $V_G := \{1, \dots, n\}$, and the edge $i \rightarrow j$ has multiplicity m_{ij} . Note that we allow arrows $(i \rightarrow j)$ and $(i \leftarrow j)$ with arbitrary non-negative multiplicities at the same time. In the example on Figure 4, one gets a circle graph (Figure 5).

Definition 3.1. A function $f : \{1, \dots, n\} \rightarrow \mathbb{Z}_{\geq 0}$ is called a G -parking function if for any non-empty subset $I \subset \{1, \dots, n\}$ there exists a vertex $i \in I$ such that the number of edges $(i \rightarrow j) \in E_G$, count with multiplicity, such that $j \notin I$ is greater than or equal to $f(i)$.

Remark 3. Note that at least one value of any G -parking function f has to be zero. Indeed, take $I = \{1, \dots, n\}$. Then, according to the definition, there exists $i \in \{1, \dots, n\}$, such that $f(i) = 0$.

Lemma 3.1. Let R be a region of an arrangement \mathcal{A} with the corresponding multigraph $G = G_{\mathcal{A}}$. Then the label λ_R is a G -parking function.

Proof. Consider the subarrangement $\mathcal{A}_R \subset \mathcal{A}$ consisting of hyperplanes of \mathcal{A} that separate the region R from the origin. Let $G_R = G_{\mathcal{A}_R} \subset G_{\mathcal{A}}$ be the corresponding sub-multigraph. The key observation is that G_R cannot contain oriented cycles. Indeed, suppose that there is a cycle $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \rightarrow i_1$. Then it follows that there exist hyperplanes $H_{i_1 i_2}^{a_1}, \dots, H_{i_k i_1}^{a_k} \in \mathcal{A}$ separating region R from the origin. Let $\mathbf{r} = (r_1, \dots, r_n) \in R$ be a point. One gets

$$r_{i_1} - r_{i_2} > a_1,$$

$$\vdots$$

$$r_{i_k} - r_{i_1} > a_k.$$

Summing the inequalities up, one gets

$$0 > a_1 + \dots + a_k,$$

which is impossible, because $a_i > 0$ for all i by construction. Contradiction.

Let us verify the G -parking function condition for λ_R . Let $I \subset \{1, \dots, n\}$ be any non-empty subset. It follows that there exists $i \in I$ such that if $i \rightarrow j$ is an edge of G_R , then $j \notin I$. Indeed, otherwise I contains an oriented cycle of G_R . Note that $\lambda_R(i)$ is equal to the outgoing index of i in G_R , and all the outgoing edges from i leave the subset I . Therefore, λ_R is a G -parking function. \square

Theorem 3.1. The labeling is surjective.

Remark 4. Note that arrangements with the same graph might have different combinatorics and even different number of regions. However, the labeling is always surjective. See Figure 6 for an example.

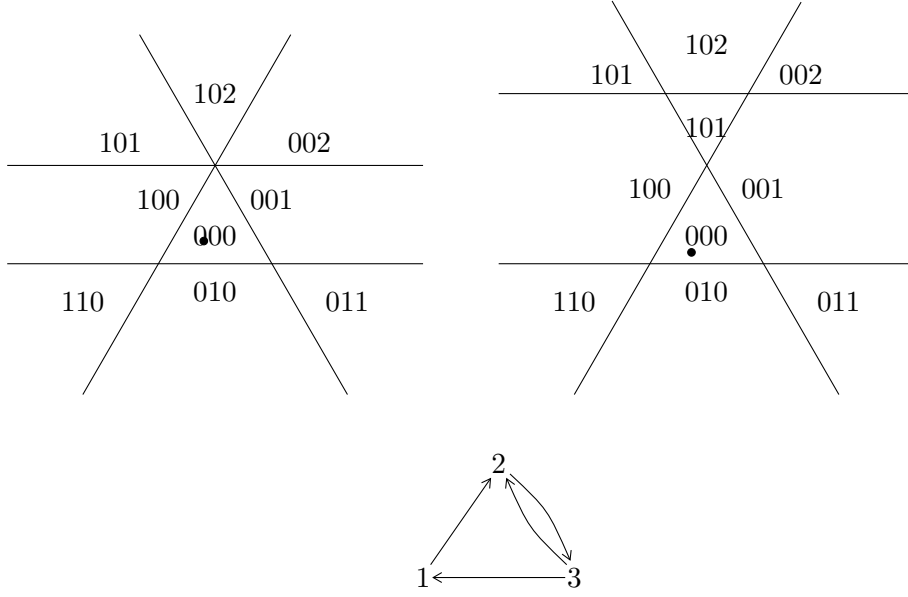


FIGURE 6. These two arrangements correspond to the same graph. Note that the labeling is surjective in both cases.

Proof of Theorem 3.1. Let $f : \{1, \dots, n\} \rightarrow \mathbb{Z}_{\geq 0}$ be a G -parking function. We want to find a region of \mathcal{A} such that the corresponding label is equal to f . We start from R_0 and cross exactly one hypersurface on each step. We proceed as follows. Suppose that we got to a region R . Split the set of indexes in two subset

$$\{1, \dots, n\} = I \sqcup J,$$

where $I := \{i : f(i) > \lambda_R(i)\}$. On our next step we cross a hyperplane of the form H_{ij}^a with $i \in I$ and $j \in J$ in positive direction, i.e. getting further away from the origin. As long as we can proceed, the following conditions will be satisfied:

- (1) For all $j \in J$ one has $f(j) = \lambda_R(j)$.
- (2) The subset $I = \{i : f(i) > \lambda_R(i)\}$ is not increasing along the way. In particular, the subset J is never empty (it was not empty in the beginning, because at least one value of f is zero).
- (3) No hyperplane of the form H_{pq}^a with $p, q \in I$ separates the region R from R_0 at any step.

Indeed, according to the algorithm, the values of the labeling can only increase along the way, and they cannot exceed $f(i)$. Condition (3) then follows from condition (2). All we need to prove now is that given these conditions, we can proceed until $I = \emptyset$ and $f = \lambda_R$. Indeed, let us start from a point $(r_1, \dots, r_n) \in R$ and move along the ray

$$x_i = r_i + \frac{t}{\#I}, \quad i \in I,$$

$$x_j = r_j - \frac{t}{\#J}, \quad j \in J,$$

$t \geq 0$. If we hit a wall of R , it has to be a hyperplane of the form H_{ij}^a with $i \in I$ and $j \in J$. If we never hit any walls, that means that all hyperplanes of that form are already between R and R_0 . In which case the subset $I \subset \{1, \dots, n\}$ violates the G -parking function condition. Indeed, since f is a G -parking function, there should exist a vertex $i \in I$ such that

$$N_R(i) := \#\{(i \rightarrow j : j \in J)\} \geq f(i).$$

However, there are no edges between vertices inside I , so we get $\lambda_R(i) = N_R(i) \geq f(i)$, which contradicts the definition of the set I . \square

Note that this result provides a simple proof of the bijectivity of the original Pak-Stanley labeling for general k . Indeed, given a k -Shi arrangement, one can change the coordinates by shifting the origin to the interior of the fundamental region. The resulting arrangement will fit into the framework of this chapter. The corresponding multigraph is the full graph with every edge having multiplicity k in each direction. The G -parking function condition specializes to the k -parking function condition.

Finally, the number of regions of a k -Shi arrangement and the number of k -parking functions are both known to be equal to $(kn + 1)^{n-1}$. Therefore, in this special case, the labeling is not only surjective but, in fact, bijective.

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